Investigations of plasma dielectric functions

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Abstract

Some general relations concerning dielectric functions can be obtained using an analysis well known from electrical network theory. On the basis of these relations a general discussion of wave stability and wave energy is presented.

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1. Introduction

The plasma dielectric function plays an important role in questions concerning:

a) The response to test charges, where the response in potential, φ (k, ω), to an applied test charge, ρ (k, ω), is given by:

$$\varphi(k,\omega) = \frac{1}{\varepsilon_0 k^2} \frac{\rho(k,\omega)}{\varepsilon_1(k,\omega) + i\,\varepsilon_2(k,\omega)} \tag{1}$$

k and ω real. ε_1 and ε_2 are the real and the imaginary part of the dielectric function respectively. This problem is relevant for wave excitations in plasmas[1,2].

b) Low-level thermal oscillations in a plasma in thermal equilibrium with a "heat bath" at a temperature T. The fluctuations of the potential are given by[3]:

$$\langle \varphi^2(k,\omega) \rangle = \frac{\varkappa T}{T \varepsilon_0 k^2} \frac{2}{\omega} \frac{\varepsilon_2(k,\omega)}{\varepsilon_1^2(k,\omega) + \varepsilon_2^2(k,\omega)} \tag{2}$$

The average can be considered as an ensemble average or a time average (ergodicity). The theorem may be generalized to the case of non-equilibrium (but stationary) plasma state[3,4]. In this case knowing ε(k, ω) is necessary, but not sufficient: we also need information on the charge-density fluctuation spectral distribution of the non-interacting particles. Ergodicity is no longer ensured and averaging must be done as ensemble averaging.

c) Specification of wave energy[5]. In this connection ε_2(k, ω) is usually neglected. Some interesting results can be obtained, however, by inclusion of this term.

In the following are given some general relations concerning ε(k, ω) which are useful in questions concerning these problems.

2. Basic Assumptions

The following assumptions will be used:

1: $\varepsilon(k,\omega) = \varepsilon^*(k,-\omega)$ i. e. $\varepsilon_1(k,\omega) = \varepsilon_1(k,-\omega)$ and $\varepsilon_2(k,\omega) = -\varepsilon_2(k,-\omega)$. (3)

II: A singularity at a finite point ω_0 on the real ω axis is of such a nature that

$$\lim_{\omega \to \omega_0} (\omega - \omega_0)\, \varepsilon(k,\omega) = 0 \tag{4}$$

This permits logarithmic singularities and branch points, but not poles on the real ω axis.

III: $\varepsilon(k, \omega)$ has no poles in the upper half of the complex ω plane.

IV: We assume $\varepsilon(k, \omega)$ to be analytic for $|\omega| \to \infty$:

$$\lim_{|\omega| \to \infty} \varepsilon(k,\omega) = A_\infty \tag{5}$$

where A_∞ may depend on k. Generally we would expect $A_\infty = 1$ since no physical system can respond to an infinitely fast oscillation. Particular models may however give $A_\infty \neq 1$. Several of the following theorems remain valid provided

$$\lim_{|\omega| \to \infty} \left| \frac{\varepsilon(k,\omega)}{\omega} \right| = 0 \tag{6}$$

a somewhat weaker condition than (5).

Apparently assumption I is the most restrictive. It permits, however, a large group of dielectric functions of physical interest. Specially we note that we have nowhere assumed the plasma to be stable: $\varepsilon(k, \omega)$ may very well have zeros in the upper half of the complex ω plane. In eq. 2 the concept of stability is introduced by assigning a temperature for the plasma.

Examples of dielectric functions satisfying the requirements above are:

$$\varepsilon(k,\omega) = 1 \div \frac{\omega_p^2}{k^2} \int_{-\infty}^{\infty} \frac{f'(v)}{v - \frac{\omega}{k}}\, dv \qquad , \qquad \mathrm{Im}\,\omega > 0$$

and by analytic continuation:

$$\varepsilon(k,\omega) = 1 \div \frac{\omega_p^2}{k^2} P\!\int_{-\infty}^{\infty} \frac{f'(v)}{v - \frac{\omega}{k}}\, dv - i\pi \frac{\omega_p^2}{k^2} f'\!\left(\frac{\omega}{k}\right), \quad \begin{matrix} \mathrm{Im}\,\omega = 0 \\ k > 0 \end{matrix} \tag{7}$$

appropriate for electron oscillations in a background of rigid ions. ω_p is the plasma frequency, and f(v) is a symmetric electron velocity distribution function such as a Maxwellian. The generalization to many component plasmas is straightforward.

When considering the ion motion we can often assume the electrons to be isothermally Boltzmann distributed at all times. We obtain

$$\varepsilon(k,\omega) = 1 - \frac{1}{(kd_e)^2} - \frac{\omega_{pi}^2}{k^2} P\int_{-\infty}^{\infty} \frac{f_i'(v)}{v - \frac{\omega}{k}}\, dv - i\pi \frac{\omega_{pi}^2}{k^2} f_i'\left(\frac{\omega}{k}\right) \qquad (8)$$

where $\omega_{p_i}^2$ is the ion plasma frequency, and d_e the electron Debye length. For a properly chosen ion velocity distribution function $f_i(v)$ (8) will satisfy our requirements. Note that A_∞ defined in (5) is a function of k in this model. We note that assumption I can be satisfied also if $f(v)$ is symmetric with respect to $v_o \neq 0$ (e.g. a drifting Maxwellian), simply by replacing ω by $\omega - kv_o$.

Also the well-known oscillator model satisfies I - IV:

$$\varepsilon(k,\omega) = 1 + \frac{1}{m} \frac{\omega_o^2 - \omega^2}{(\omega_o^2 - \omega^2)^2 + g^2\omega^2} + i\frac{1}{m} \frac{\omega g}{(\omega_o^2 - \omega^2)^2 + g^2\omega^2} \qquad (9)$$

obtained on the basis of the equation

$$m\ddot{x} + m\omega_o^2 x + g m \dot{x} = 0 \qquad (10)$$

where ω_o accounts for the undamped oscillation and g for the damping (friction). For free electrons $\omega_o = 0$.

Finally is mentioned a condition which, for physical reasons, should be imposed on $\varepsilon(k,\omega)$:

$$\left\langle \left(\frac{\partial^n \varphi(k,t)}{\partial t^n} \right)^2 \right\rangle = 2 \frac{\varkappa T}{\pi \varepsilon_o k^2} \int_0^{\infty} \omega^{2n-1} \frac{\varepsilon_2(k,\omega)}{\varepsilon_1^2(k,\omega) + \varepsilon_2^2(k,\omega)} d\omega \qquad (11)$$

must be finite for all n ((11) can be derived directly from (2)). This means that $\varepsilon_2(k,\omega)$ must vanish faster than any power of ω, e.g. exponentially[6]) (the denominator is a constant at high frequencies).

It is worth noting that (7) and (8) fulfil this requirement, while the simple model (9) does not. The obvious reason is that (10) is insufficient at high frequencies. We have for instance neglected the radiation damping of an accelerated charged particle

$$\frac{2}{3} \frac{e^2}{4\pi\varepsilon_o} \frac{1}{c^3} \dddot{x} \qquad (12)$$

and we have also assumed g to be a constant independent of ω. It can be shown[7] that this is only a valid assumption as long as the duration of a collision between an electron and the damping component is vanishing compared with g^{-1}.

We can generalize (11) so that

$$\left\langle\left(\frac{\partial^{n+m}\varphi(x,t)}{\partial x^m \partial t^n}\right)^2\right\rangle = \frac{\varkappa T}{\pi^2 \varepsilon_0} \iint k^{2m-2}\, \omega^{2n-1} \frac{\varepsilon_2(k,\omega)}{\varepsilon_1^2(k,\omega)+\varepsilon_2^2(k,\omega)}\, d\omega\, dk \tag{11a}$$

The k-integral will generally be divergent for large m. This is not surprising since our theory (generally a mean-field theory) breaks down for very large k (small wavelengths).

We consider k as a fixed non-zero quantity throughout. The point $(k, \omega) = (0, 0)$ will often exhibit singularities. The k-dependence will only give problems when evaluating integrals of the type:

$$\int \frac{\partial \varepsilon}{\partial \omega}\, d\omega \quad .$$

In general we expect $\varepsilon(k, \omega)$ to be an even function of k.

3. Some General Relations

Assumptions II-IV ensure that $\varepsilon(k,\omega)$ satisfies the Kronig - Kramers relations[5, 8)]

$$\varepsilon_1(k,\omega) - A_\infty = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\varepsilon_2(k,\gamma)}{\gamma - \omega}\, d\gamma \tag{13}$$

$$\varepsilon_2(k,\omega) = -\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\varepsilon_1(k,\gamma) - A_\infty}{\gamma - \omega}\, d\gamma \tag{14}$$

Examples (7), (8), and (9) satisfy these relations.

Using assumption I, (13) and (14) can be reduced to

$$\varepsilon_1(k,\omega) - A_\infty = \frac{2}{\pi} P\int_0^{\infty} \frac{\gamma\, \varepsilon_2(k,\gamma)}{\gamma^2 - \omega^2}\, d\gamma$$

$$\varepsilon_2(k,\omega) = -\frac{2\omega}{\pi} P\int_0^{\infty} \frac{\varepsilon_1(k,\gamma) - A_\infty}{\gamma^2 - \omega^2}\, d\gamma$$

We omit the principal value sign, P, by writing

$$\varepsilon_1(k,\omega) - A_\infty = \frac{2}{\pi}\int_0^\infty \frac{\gamma\,\varepsilon_2(k,\gamma) - \omega\,\varepsilon_2(k,\omega)}{\gamma^2 - \omega^2}\,d\gamma \qquad (15)$$

$$\varepsilon_2(k,\omega) = -\frac{2\omega}{\pi}\int_0^\infty \frac{\varepsilon_1(k,\gamma) - \varepsilon_1(k,\omega)}{\gamma^2 - \omega^2}\,d\gamma \qquad (16)$$

Contrary to (13) and (14) these relations offer no difficulties when differentiated with respect to ω.

Introducing

$$u = \ln\left(\frac{\gamma}{\omega}\right) \quad , \quad \frac{\gamma}{\omega} = \exp u \quad , \quad d\gamma = \gamma\, du$$

we get

$$\varepsilon_1(k,\omega) - A_\infty = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\exp u\; \varepsilon_2(k,u) - \varepsilon_2(k,\omega)}{\sinh u}\, du$$

Using $(\sinh u)^{-1}\, du = -d \ln \coth |u/2|$ we get by partial integration

$$\varepsilon_1(k,\omega) - A_\infty = \frac{1}{\pi}\int_{-\infty}^{\infty} \ln\coth\left|\frac{u}{2}\right| \frac{\partial \exp u\; \varepsilon_2(k,u)}{\partial u}\, du \qquad (17)$$

since $\ln\coth(u/2) \to \exp(-u)$ for $u \to \infty$ and $\varepsilon_2(k,\omega) \to 0$ for $\omega \to \infty$.

By reintroducing ω we obtain

$$\varepsilon_1(k,\omega) - A_\infty = \frac{1}{\omega\pi}\int_0^\infty \ln\left|\frac{\gamma+\omega}{\gamma-\omega}\right| \frac{\partial\,\gamma\,\varepsilon_2(k,\gamma)}{\partial\gamma}\, d\gamma \qquad (18)$$

$\varepsilon_1(k,\omega)$ thus depends on $\dfrac{\partial\,\gamma\,\varepsilon_2(k,\gamma)}{\partial\gamma}$ at all frequencies multiplied by a weight function $W(\gamma) = \ln|(\gamma+\omega)/(\gamma-\omega)|$. $W(\gamma)$ has the following properties:

for $\gamma = \omega(1+\xi)$; $|\xi| \ll 1$; $W(\gamma) \simeq \ln 2/|\xi|$;

for $\gamma \gg \omega$; $W(\gamma) \simeq 1\,\frac{\omega}{\gamma}$;

for $\gamma \ll \omega$; $W(\gamma) \simeq 2\,\gamma/\omega$,

specially $\omega = 0$; $W(\gamma) = 0$.

$W(\gamma) \geq 1$ in the interval

$[\,\omega(e+1)/(e-1);\ \omega(e-1)/(e+1)\,] \sim [0.46\,\omega;\ 2.16\,\omega\,]$

and $W(Y) < 1$ elsewhere. Returning to (17) we write:

$$\varepsilon_1(k,\omega) - A_\infty = \frac{\pi}{2}\frac{\partial\omega\varepsilon_2(k,\omega)}{\partial\omega} + \frac{1}{\pi}\int_{-\infty}^{\infty}\left\{\frac{\partial\exp u\,\varepsilon_2(k,u)}{\partial u} - \frac{\partial\omega\varepsilon_2(k,\omega)}{\partial\omega}\right\}\ln\coth\left|\frac{u}{2}\right|du \tag{19}$$

using the fact that $\frac{\partial}{\partial u}\left[\exp(u)\,\varepsilon_2(k,u)\right]_{u=0} = \frac{\partial}{\partial\omega}\,\omega\varepsilon_2(k,\omega)$ and $\int_{-\infty}^{\infty}\ln\coth|u/2|\,du = \pi^2/2$. The integrand in (19) is zero for $u = 0$, i.e. at the point where the weight function $G(u) = \ln\coth|u/2|$ is sharply peaked. $G(u) > 1$ in the interval $\sim[-0.8;\ 0.8]$. For large $|u|$, $G(u) \sim \exp(-|u|)$. If $\exp(u)\varepsilon_2(k,u)$ varies slowly around $u = 0$, we may neglect the integral term and approximate $\varepsilon_1(k,\omega) - A_\infty \approx \frac{\pi}{2}\frac{\partial}{\partial\omega}(\omega\varepsilon_2(k,\omega))$.

W(u) and G(u) are shown on figs. 1 and 2.

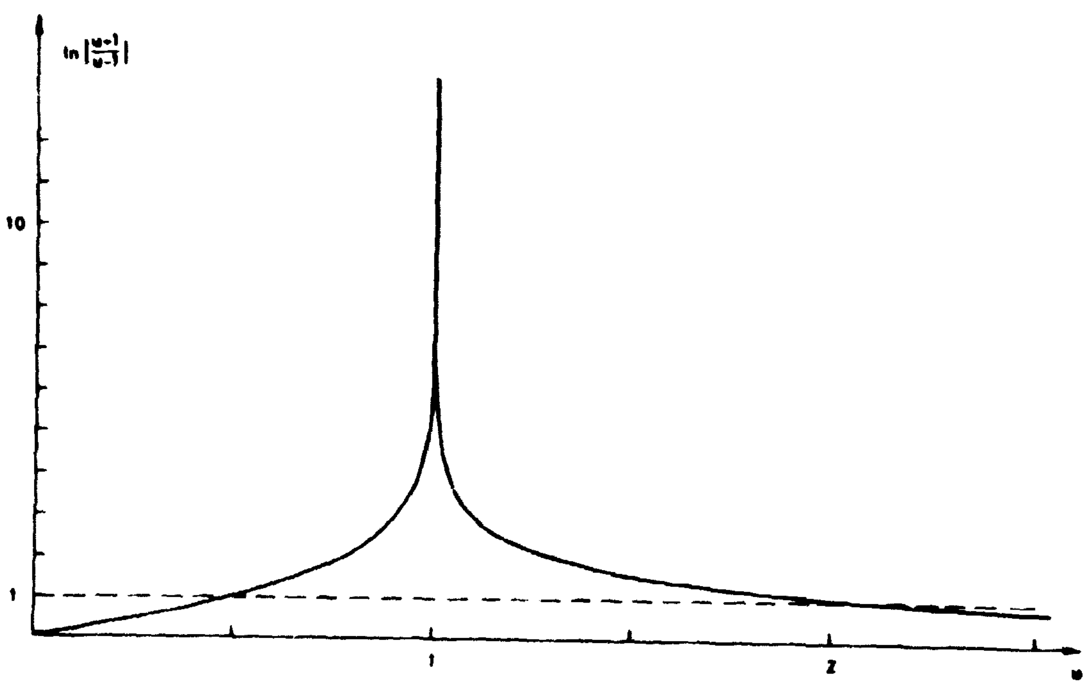

Fig. 1

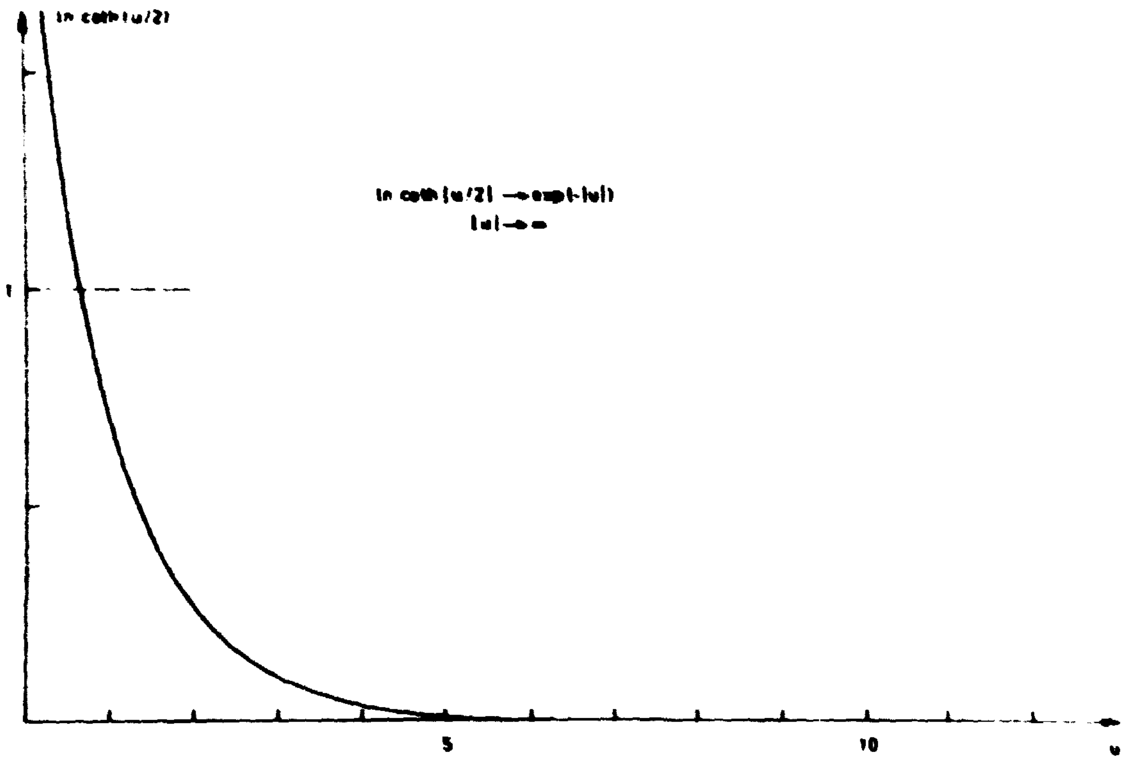

Fig. 2

An alternative relation between ϵ_1 and ϵ_2 can be obtained. We introduce the following power series expansion for small ω:

$$\mathcal{E}(k,\omega) = A_0 + i B_0 \omega + C_0 \omega^2 + i D_0 \omega^3 + \cdots \tag{20}$$

using $\epsilon_2(k, \omega = 0) = 0$ and the symmetry properties of $\epsilon(k, \omega)$.

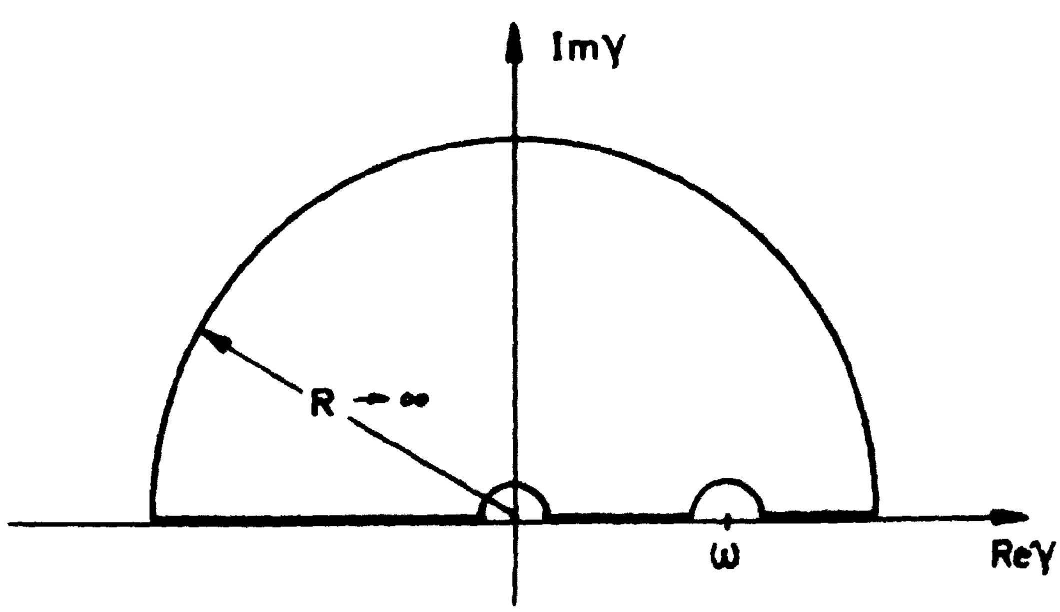

Fig. 3

Integrating $(\varepsilon(k,\gamma)-A_\infty)/\gamma(\gamma-\omega)$ along the path shown on fig. 3, we obtain an analogue to the Kronig - Kramers relations:

$$\varepsilon_2(k,\omega) = -\frac{\omega}{\pi} P\int_{-\infty}^{\infty} \frac{\varepsilon_1(k,\gamma)-A_0}{\gamma(\gamma-\omega)}\,d\gamma \tag{21}$$

$$\varepsilon_1(k,\omega)-A_0 = \frac{\omega}{\pi} P\int_{-\infty}^{\infty} \frac{\varepsilon_2(k,\gamma)}{\gamma(\gamma-\omega)}\,d\gamma \tag{22}$$

Using the symmetry properties of $\varepsilon(k,\omega)$, (assumption I) (21) reduces to (16), while (22) reduces to:

$$\varepsilon_1(k,\omega)-A_0 = \frac{2\omega^2}{\pi} P\int_0^{\infty} \frac{\varepsilon_2(k,\gamma)}{\gamma(\gamma^2-\omega^2)}\,d\gamma \tag{23}$$

or

$$\varepsilon_1(k,\omega)-A_0 = \frac{2\omega^2}{\pi} \int_0^{\infty} \frac{\varepsilon_2(k,\gamma)-\varepsilon_2(k,\omega)}{\gamma(\gamma^2-\omega^2)}\,d\gamma \tag{24}$$

Using the variable transformation introduced previously we obtain:

$$\varepsilon_1(k,\omega)-A_0 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\partial\, \exp(-u)\,\varepsilon_2(k,u)}{\partial u}\, \ln\coth\left|\frac{u}{2}\right| du =$$

$$= \frac{\pi}{2}\omega^2 \frac{\partial\left(\frac{\varepsilon_2(k,\omega)}{\omega}\right)}{\partial\omega} + \frac{1}{\pi}\int_{-\infty}^{\infty}\left\{\frac{\partial \exp(-u)\,\varepsilon_2(k,u)}{\partial u} - \omega^2\frac{\partial\left(\frac{\varepsilon_2(k,\omega)}{\omega}\right)}{\partial\omega}\right\} \ln\coth\left|\frac{u}{2}\right| du$$

or

$$\varepsilon_1(k,\omega)-A_0 = \frac{\omega}{\pi}\int_0^{\infty} \ln\left|\frac{\gamma+\omega}{\gamma-\omega}\right| \frac{\partial\left(\frac{\varepsilon_2(k,\gamma)}{\gamma}\right)}{\partial\gamma}\,d\gamma \tag{25}$$

Similarly we obtain the formula

$$\varepsilon_2(k,\omega) = -\frac{1}{\pi}\int_0^{\infty} \ln\left|\frac{\gamma+\omega}{\gamma-\omega}\right| \frac{\partial\,\varepsilon_1(k,\gamma)}{\partial\gamma}\,d\gamma \tag{26}$$

or

$$\varepsilon_2(k,\omega) = -\frac{\pi}{2}\omega\frac{\partial\varepsilon_1(k,\omega)}{\partial\omega} - \frac{1}{\pi}\int_{-\infty}^{\infty}\left\{\frac{\partial\varepsilon_1(k,u)}{\partial u} - \omega\frac{\partial\varepsilon_1(k,\omega)}{\partial\omega}\right\}\ln\coth\left|\frac{u}{2}\right| du \tag{27}$$

where we have introduced $\gamma = \omega \exp u$, as before. Evidently ε_2 depends on

$\frac{\partial \varepsilon_1(k,\gamma)}{\partial \gamma}$ at all frequencies multiplied by the weight function $W(\gamma)$.

If we now assume $\varepsilon_2 \geq 0$, as it is for instance for media in thermal equilibrium (dissipation at all frequencies), then the integral in (26) must be negative for all ω.

Generally $\frac{\partial \varepsilon_1}{\partial \omega}$ will have both negative and positive values. Eq. 26 then shows that ε_2 generally obtains its maximum value at a point ω where $\frac{\partial \varepsilon_1}{\partial \omega} < 0$, and its minimum value ($\varepsilon_2 \sim 0$) where $\frac{\partial \varepsilon_1}{\partial \omega} \sim 0$ or $\frac{\partial \varepsilon_1}{\partial \omega} > 0$. The following situation may however arise: if there are two intervals on the ω axis where $\frac{\partial \varepsilon_1}{\partial \omega} < 0$ separated by a narrow region (narrow compared with the variation length of $W(\gamma)$) where $\frac{\partial \varepsilon_1}{\partial \omega} \geq 0$, then ε_2 may reach its maximum value in this region.

We can get a relation for $\varepsilon_1(k, \omega = 0) = A_0$ using (15):

$$A_0 - A_\infty = \frac{2}{\pi} \int_0^\infty \varepsilon_2(k,\gamma) \frac{d\gamma}{\gamma} \tag{28}$$

or if we measure the frequency on a logarithmic scale

$$A_0 - A_\infty = \frac{2}{\pi} \int_{-\infty}^{\infty} \varepsilon_2(k,u)\, du \tag{29}$$

If $\varepsilon_2 \geq 0$, this relation obviously means that $A_0 > A_\infty$. A similar relation can be obtained considering $\lim_{\omega\to\infty} (\omega\varepsilon_2)$ and using (16).

$$B_\infty = \frac{2}{\pi} \int_0^\infty [\varepsilon_1(k,\gamma) - A_\infty]\, d\gamma \tag{30}$$

where $B_\infty = \lim_{\omega\to\infty} [\omega\varepsilon_2(k,\omega)]$. As mentioned earlier $B_\infty = 0$ in general. Apparently the integrals in (29) and (30) are sensitive only to the change of the behaviour of $\varepsilon(k,\omega)$ at $\omega = 0$ and $\omega \sim \infty$, and not to the curve shape itself.

A large number of related formulae can be obtained by integrating $(\varepsilon(k,\omega) - A_\infty)^2$, $(\varepsilon(k,\omega) - A_0)^2/\omega^2$, $(\varepsilon(k,\omega) - A_\infty)^3$ etc. along the path shown in fig. 3. See ref. 9. These relations are useful when determining the coefficients in the series expansion (20).

One of these relations, however, is particularly interesting. Integrating $\varepsilon^2(k,\omega)/\omega$ along the path mentioned we obtain:

$$\int_0^\infty \varepsilon_1(k,\gamma)\varepsilon_2(k,\gamma)\frac{d\gamma}{\gamma} = \frac{\pi}{4}(A_0^2 - A_\infty^2)$$

$$= \frac{\pi}{4}(A_0 + A_\infty)(A_0 - A_\infty) \qquad (31)$$

Using (28):

$$\int_0^\infty \varepsilon_1(k,\gamma)\varepsilon_2(k,\gamma)\frac{d\gamma}{\gamma} = \frac{1}{2}(A_0 + A_\infty)\int_0^\infty \varepsilon_2(k,\gamma)\frac{d\gamma}{\gamma}$$

$$\int_0^\infty [\varepsilon_1(k,\gamma) - E]\,\varepsilon_2(k,\gamma)\frac{d\gamma}{\gamma} = 0$$

where $E = 1/2\,(A_\infty + A_0)$ can be considered as a "reference value" for ε_1. On a logarithmic frequency scale we obtain

$$\int_{-\infty}^\infty [\varepsilon_1(k,u) - E]\,\varepsilon_2(k,u)\,du = 0 \qquad (32)$$

This relation states that ε_1 and ε_2 are orthogonal on a logarithmic frequency scale if we measure ε_1 from its reference value. We note that many of the relations in this section indicate that the use of a logarithmic frequency scale is preferable. ((32) is a consequence of the symmetry properties of $\varepsilon(k,\omega)$).

Finally we give a series expansion for $\varepsilon_1(k,\omega) - A_\infty$ for large ω ((20) was valid for small ω). Using (13) and the relation $(\gamma-\omega)^{-1} = -\frac{1}{\omega}\sum_{n=0}^{\infty}(\gamma/\omega)^n$ we obtain

$$\varepsilon_1(k,\omega) - A_\infty = -\frac{1}{\pi\omega}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\varepsilon_2(k,\gamma)\left(\frac{\gamma}{\omega}\right)^n d\gamma \qquad (33a)$$

$= 0$ for n even or $n = 0$

$$= \frac{-2}{\pi\omega}\sum_{n=1}^{\infty}\int_0^{\infty}\varepsilon_2(k,\gamma)\left(\frac{\gamma}{\omega}\right)^n d\gamma \qquad \text{for } n \text{ uneven}$$

or

$$\varepsilon_1(k,\omega) - A_\infty = -\frac{2}{\pi\omega}\sum_{n=0}^{\infty}\int_0^{\infty}\varepsilon_2(k,\gamma)\left(\frac{\gamma}{\omega}\right)^{2n+1} d\gamma \qquad (33b)$$

valid for sufficiently large ω provided that $\int_0^{\infty} \varepsilon_2(k,\gamma)\gamma^{2n+1}\,d\gamma$ converges. As mentioned earlier a physical argument indicates that this is the case. For sufficiently large ω we need to consider only the term for n = 0 in (33 b):

$$\varepsilon_1(k,\omega) - A_\infty \simeq -\frac{2}{\pi\omega^2}\int_0^\infty \varepsilon_2(k,\gamma)\gamma\, d\gamma \tag{34}$$

so

$$\frac{\partial\,\varepsilon_1(k,\omega)}{\partial\omega} \simeq \frac{4}{\pi\omega^3}\int_0^\infty \varepsilon_2(k,\gamma)\gamma\, d\gamma \tag{35}$$

Provided the integral converges, we conclude that $\varepsilon_1 < A_\infty$ and $\frac{\partial\varepsilon_1}{\partial\omega} > 0$ for large ω. (As before we assume $\varepsilon_2 \geq 0$). The following example will show that we can very well have an $\varepsilon(k,\omega)$ satisfying assumptions I - IV which violates the convergence criterion for (33). Assume that $\varepsilon_1 = 1 + \exp(-\omega^2)$. Then $\varepsilon_2 = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \exp(-\gamma^2)/(\gamma-\omega)\,d\gamma$. We can introduce the plasma dispersion function $Z(\omega)$. See ref. 10. Then $\varepsilon_2 = -\frac{\sqrt{\pi}}{\pi}\mathrm{Re}\,Z(\omega)$ and $\varepsilon_2 \sim \omega^{-1}$ for large ω. The integral in (33) diverges, but $\varepsilon(k,\omega)$ satisfies our assumptions. Note that $\frac{\partial\varepsilon_1}{\partial\omega} < 0$ for all ω in this case so (26) is satisfied. A general relation for $\frac{\partial\varepsilon_1}{\partial\omega}$ can be obtained by differentiating (13) or (15)

$$\frac{\partial\varepsilon_1(k,\omega)}{\partial\omega} = \frac{4}{\pi}\int_0^\infty \frac{(\gamma^2-\omega^2)\left(\omega\frac{\partial\varepsilon_2(k,\omega)}{\partial\omega}+\varepsilon_2(\omega,k)\right)+2\omega\left(\gamma\varepsilon_2(k,\gamma)-\omega\varepsilon_2(k,\omega)\right)}{(\gamma^2-\omega^2)^2}\,d\gamma \tag{36}$$

If $\varepsilon_2(k,\omega) = 0$ and $\frac{\partial\varepsilon_2}{\partial\omega} = 0$ for some ω, we obtain the formula shown in ref. 5. (Note that if $\varepsilon_2 \geq 0$ and differentiable, then $\varepsilon_2 = 0$ implies $\frac{\partial\varepsilon_2}{\partial\omega} = 0$). This relation can be used also if ε_2 and $\frac{\partial\varepsilon_2}{\partial\omega} > 0$. In the same frequency range we obtain

$$\frac{\partial\left[\omega^2(\varepsilon_1(k,\omega)-A_\infty\right]}{\partial\omega} \simeq \frac{4\omega}{\pi}\int_0^\infty \frac{\gamma^2\varepsilon_2(k,\gamma)}{(\gamma^2-\omega^2)^2}\,d\gamma \tag{37}$$

i.e.

$$\frac{\partial\,\varepsilon_1(k,\omega)}{\partial\omega} > \frac{2}{\omega}\left(A_\infty - \varepsilon_1(k,\omega)\right)$$

if $\varepsilon_1 < A_\infty$ (as it is for instance for large ω, generally); this inequality is more stringent than $\frac{\partial\varepsilon_1}{\partial\omega} > 0$ [5]. These relations are of particular interest in the case where $f(v)$ in (7) and (8) has a plateau, i. e. $f'(\omega/k) = 0$ for some ω, but $f'(\omega/k) < 0$ elsewhere.

A relation for $\varepsilon_2(k,\omega)$ similar to (34) can be obtained from (14)

$$\varepsilon_2(k,\omega) = \frac{1}{\pi\omega} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} [\varepsilon_1(k,\gamma) - A_\infty]\left(\frac{\gamma}{\omega}\right)^n d\gamma \tag{38}$$

Using $\varepsilon_1(k,\omega) = \varepsilon_1(k,-\omega)$ (38) reduces to

$$\varepsilon_2(k,\omega) = \frac{2}{\pi\omega} \sum_{n=0}^{\infty} \int_{0}^{\infty} [\varepsilon_1(k,\gamma) - A_\infty]\left(\frac{\gamma}{\omega}\right)^{2n} d\gamma \tag{39}$$

However, if $\varepsilon_2(k,\omega)$ vanishes exponentially for large ω, then all the integrals in (39) are zero.

Using (16) we obtain

$$\frac{\partial \varepsilon_2(k,\omega)}{\partial \omega} = -\frac{2}{\pi} \int_0^{\infty} \frac{(\gamma^2-\omega^2)\left(\varepsilon_1(k,\gamma) - \dfrac{\partial \omega\, \varepsilon_1(k,\omega)}{\partial \omega}\right) + 2\omega^2(\varepsilon_1(k,\gamma) - \varepsilon_1(k,\omega))}{(\gamma^2-\omega^2)^2} d\gamma$$

or

$$\frac{\partial \varepsilon_2(k,\omega)}{\partial \omega} = -\frac{2}{\pi} \int_0^{\infty} \frac{(\omega^2+\gamma^2)(\varepsilon_1(k,\gamma) - \varepsilon_1(k,\omega)) + \omega \dfrac{\partial \varepsilon_1(k,\omega)}{\partial \omega}(\omega^2-\gamma^2)}{(\gamma^2-\omega^2)^2} d\gamma \tag{40}$$

If ε_1 is differentiable and $\varepsilon_1(\Omega)$ is its maximum value, then $\frac{\partial \varepsilon_2}{\partial \Omega} < 0$ since $\frac{\partial \varepsilon_1}{\partial \Omega} = 0$ and $\varepsilon_1(k,\gamma) - \varepsilon_1(k,\Omega) < 0$ for all γ. If $\varepsilon_1(\Omega)$ is the minimum value, then $\frac{\partial \varepsilon_2}{\partial \Omega} > 0$ by a similar argument. Specially for $\omega = 0$ we have

$$\left(\frac{\partial \varepsilon_2(k,\omega)}{\partial \omega}\right)_{\omega=0} = -\frac{2}{\pi}\int_0^{\infty} [\varepsilon_1(k,\gamma) - \varepsilon_1(k,0)] \frac{d\gamma}{\gamma^2}$$

This integral is negative if the medium is in equilibrium ($\varepsilon_2 \geq 0$) since $\varepsilon_2(k,0) = 0$.

As mentioned earlier in many cases we are interested in $\varepsilon(k,\omega)^{-1}$ rather than $\varepsilon(k,\omega)$. (See (1) and (2)). If $\varepsilon(k,\omega)$ has no zeros in the upper half of the comples ω plane, we can substitute ε^{-1} for ε in all the relations in this section.

Calculations related to those shown in this section are well known from the theory of electrical network analysis[9].

4. Extension to Complex ω

The relations shown in the preceding sections are valid for real ω and k. Special interest is taken, however, in solutions to $\varepsilon(k,\omega) = 0$ where $\omega = \Omega_k + \gamma_k$. Generally it is assumed that $|\gamma_k| \ll \Omega_k$, otherwise it is hardly reasonable to talk about a wave-like solution at all. Looking for a zero close to the real ω-axis we take into account only the first term in a Taylor expansion[11].

$$\varepsilon(k,\omega) = 0 = \varepsilon_1(k,\Omega_k + i\gamma_k) + i\varepsilon_2(k,\Omega_k + i\gamma_k)$$

$$\simeq \varepsilon_1(k,\Omega_k) + i\varepsilon_2(k,\Omega_k) - \gamma_k \frac{\partial \varepsilon_2}{\partial \Omega_k} + i\gamma_k \frac{\partial \varepsilon_1}{\partial \Omega_k}$$

We assume ε_2 and $\frac{\partial \varepsilon_2}{\partial \Omega_k}$ to be small. The third term is then negligible (since γ_k is small also), and we obtain

$$\varepsilon_1(k,\Omega_k) \simeq 0 \qquad \text{and} \qquad \gamma_k \simeq - \frac{\varepsilon_2(k,\Omega_k)}{\dfrac{\partial \varepsilon_1(k,\Omega_k)}{\partial \Omega_k}} \tag{41}$$

γ_k may be negative or positive corresponding to damped or growing oscillations respectively.

The contribution to (1) and (2) will be dominated by the weakly damped modes. As mentioned earlier (2) is, strictly speaking, only meaningful if f(v) is a Maxwellian with a thermal spread corresponding to T. In this case the solution of (41) is standard:

$$\Omega_k \simeq \omega_p + \frac{3}{2}\frac{\varkappa T}{m\omega_p} k^2 \quad , \quad \gamma_k \simeq -\sqrt{\frac{\pi}{8}}\, \frac{\omega_p}{(dk)^3} \exp\left(-\frac{1}{2(dk)^2}\right)$$

where d is the Debye length.

Obviously the information gained in the preceding section is useful also when evaluating (41).

If $\varepsilon_2(k,\omega) \simeq 0$ for $\omega \simeq 0$, then the system must be stable, i.e. $\gamma_k < 0$. In the previous section we showed that the assumption of small ε_2 and $\frac{\partial \varepsilon_2}{\partial \omega}$ implied $\frac{\partial \varepsilon_1}{\partial \omega} > 0$, so $\gamma_k < 0$. Further we note that according to (41) the assumptions of small $|\gamma_k|$ and ε_2 are consistent. If also $\frac{\partial \varepsilon_1}{\partial \Omega_k}$ is small, we must include more terms in the Taylor expansion. We have seen an example where $\frac{\partial \varepsilon_1}{\partial \omega} < 0$ for all ω, but in this case $\varepsilon_1 > 1$ for all ω so (41) cannot be satisfied. In some cases, aperiodic motion may be of interest ($\omega \simeq i\gamma_k$), particularly when f(v) has a local minimum for v = 0, i.e. a two-stream-like situation[12]. Using $\varepsilon_2(k,0) = 0$ and $\frac{\partial \varepsilon_1}{\partial \omega}_{\omega=0}$, we obtain

$$\Omega_k \simeq 0 \qquad , \qquad \gamma_k \simeq \frac{\varepsilon_1(k,0)}{\left.\frac{\partial \varepsilon_2(k,\omega)}{\partial \omega}\right|_{\omega=0}}$$

provided that ε_1 is small. This necessarily implies that $\varepsilon_2 < 0$ for some frequencies according to (28), since A_∞ is unaffected by changes in f(v).

Obviously the existence of undamped oscillations (Ω, k) requires both $\varepsilon_1(k,\Omega) = 0$ and $\varepsilon_2(k,\Omega) = 0$. Using (7) or (8) we assume that $\varepsilon_2(k,\Omega) = 0$ for some (k,Ω), i.e. the distribution function has a plateau or local extrenum for $|v| = \frac{\Omega}{k}$. The requirement $\varepsilon_1(k,\Omega) = 0$ then imposes the following condition on $\varepsilon_2(k,\omega)$:

$$A_\infty = \frac{2}{\pi}\int_0^\infty \frac{\gamma\,\varepsilon_2(k,\gamma)}{\Omega^2-\gamma^2}\,d\gamma \tag{42}$$

We have used (15). Using (29) we alternatively obtain

$$A_0 = \frac{2\Omega^2}{\pi}\int_0^\infty \frac{\varepsilon_2(k,\gamma)}{\gamma(\Omega^2-\gamma^2)}\,d\gamma \tag{43}$$

The difference between (43) and (42) satisfies the general relation (28) as it should. Equations 42 and 43 are necessary and sufficient conditions for marginal stability although not as general as the Penrose criterion[13] because of our assumption I. On the other hand (42) and (43) can also be applied to the case where f(v) has a plateau, while the Penrose criterion requires a local minimum. In order to make (42) and (43) appear more familiar we can insert f'(v). For instance using (7), (42) becomes:

$$A_\infty = \frac{2\omega_p^2}{k^2}\int_0^\infty \frac{\gamma\, f'(\frac{\gamma}{k})}{\gamma^2-\Omega^2}\,d\gamma \tag{42a}$$

or

$$A_\infty = \frac{2\omega_e^2}{k^2}\int_0^\infty \frac{\xi f'(\xi)}{\xi^2 - (\frac{\Omega}{k})^2} d\xi \tag{42b}$$

Using (13) we can obtain the Penrose criterion as follows: Assume that for certain (k, Ω) we have $\epsilon_1(k,\Omega) = \epsilon_2(k,\Omega) = 0$. Then the following relation holds:

$$A_\infty = 1 = \frac{\omega_e^2}{k^2}\int_{-\infty}^{\infty} \frac{f'(\frac{\gamma}{k})}{\gamma - \Omega} d\gamma = \frac{\omega_e^2}{k^2}\int_{-\infty}^{\infty} \frac{f'(\xi)}{\xi - \frac{\Omega}{k}} d\xi$$

$$= \frac{\omega_e^2}{k^2}\int_{-\infty}^{\infty} \frac{d\left[f(\xi) - f(\frac{\Omega}{k})\right]}{\xi - \frac{\Omega}{k}}$$

We omit the principal value sign since $f'(\Omega/k) = 0$ when $\epsilon_2(k,\Omega) = 0$. Conversely we can consider this equality as a condition for marginal stability: if the integral is positive for some Ω/k, then the equality can be fulfilled for a suitable k-value. By partial integration we obtain the Penrose criterion for marginal stability:

$$\int_{-\infty}^{\infty} \frac{f(\xi) - f(\frac{\Omega}{k})}{(\xi - \frac{\Omega}{k})^2} d\xi > 0$$

Apparently this criterion applies also to the case where f(v) has a plateau for $v = \frac{\Omega}{k}$.

These criteria are exact (but linear), while (41) is useful for a rough estimate.

Equation 41 is, however, misleading in one sense: it indicates that Landau damping and growth are complementary processes, an assumption which gives rise to the physical explanation which predicts damping when there are more particles with velocities slightly below the phase velocity ω/k than there are particles with velocities slightly above ω/k, i.e. $f'(\omega/k) < 0$. Conversely amplification is expected when $f'(\omega/k) > 0$. Adopting the nomenclature of van Kampen and Case[14,15] we find normal mode solutions to

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} + \frac{e}{m}\nabla\varphi\,\frac{df(v)}{dv} = 0$$

$$\nabla^2\varphi = \frac{e}{\varepsilon_0} n_1 \equiv \frac{e}{\varepsilon_0}\int_{-\infty}^{\infty} f_1\,dv$$

of the form $f_1 = f^{k,\omega}\exp(i(kx-\omega t))$, $n_1 = n^{k,\omega}\exp(i(kx-\omega t))$, ω and k real and

$$f^{k,\omega} = \frac{\omega_p^2}{k^2} P\frac{f'(v)}{v-\frac{\omega}{k}} + \left(1 - \frac{\omega_p^2}{k^2} P\int_{-\infty}^{\infty}\frac{f'(v)}{v-\frac{\omega}{k}}dv\right)\delta\left(v-\frac{\omega}{k}\right) \quad (44)$$

while we normalize $n^{k,\omega} = 1$. We are considering one dimension for simplicity only. For a stable plasma the evolution of an arbitrary initial perturbation will be completely described by a superposition of these modes:

$$f_1(x,v,t) = \iint C(u,k)\, f^{k,u}\exp(ik(x-ut))\,dk\,du \quad (45)$$

where C(u, k) is a suitable weight function[14,15,16]. $u = \frac{\omega}{k}$ is introduced for convenience. If the plasma is unstable, we must add a discrete set of modes for the right-hand side of (45):

$$\sum_j \int \exp(ik(x-u_j^k t))\, f_j^{k,u_j^k}\,dk \quad (46)$$

where u_j^k is given as the solution to

$$1 - \frac{\omega_p^2}{k^2}\int\frac{f'(v)}{v-u_j^k}dv = 0 \quad (47)$$

and

$$f_j^{k,u_j^k} = \frac{\omega_p^2}{k^2}\,\frac{f'(v)}{v-u_j^k} \quad (48)$$

Note that if u_j^k is a solution to (47), then $(u_j^k)^*$ is a solution also. The roots, u_j^k, are assumed to be simple. An equation describing the evolution of the density is obtained by integration with respect to v, using the normalization $n^{\omega,k} = 1$. The contribution from (45) is damped in time according to Riemann's theorem. The familiar Landau damping thus appears as phase mixing of van Kampen modes. Obviously (considering (45)) it does not make sense to speak of the phase velocity of a wave at all. When experimenters nevertheless measure a phase velocity this is due to the fact that C(u, k) usually has a pronounced maximum which accentuates a certain phase velocity, u. The existence of such maxima depends, however, both on the background distribution f(v) and the perturbation g(v)[15].

The contribution from (46) shows an entirely different behaviour. Each term in the sum obeys a dispersion relation $u_j = u_j(k)$ determined by (47). Apparently only positive γ_k in (41) can characterize normal modes: negative γ_k can only account for the influence of the background plasma on the damping. As mentioned before this is not sufficient information; if we choose the velocity distribution g(v) in the perturbation as a single van Kampen mode, then the perturbation will not be damped at all. In most experiments $g(v) \simeq f(v)$, and then (41) is a good approximation.

If a physical explanation in terms of first order wave - particle interaction is adopted in order to explain the damping of (45), then it seems unreasonable to explain the growth, given by (46), by wave - particle interaction also. A possible alternative explanation is instability due to pair production of waves with positive and negative energy. As shown by Fukai and Harris[17], from a quantummechanical approach, this is the explanation of the similar multi beam problem, and a generalization seems reasonable. Of course the energy source is the group of fast particles, this explanation concerns only the mechanism of energy transfer.

5. Field Energy in Dispersive Media

The rate of change of electric energy in a unit volume of the plasma is

$$\frac{dU}{dt} = \frac{1}{2} \underline{E} \cdot \frac{d\underline{D}}{dt} \tag{49}$$

where $\underline{D} = \varepsilon_0 \underline{\underline{\varepsilon}} \cdot \underline{E}$. For simplicity we assume that ε is a scalar function. The relations in the previous sections are valid for each element if ε is a tensor. The k-dependence is unimportant for the arguments in this section, and it is therefore neglected. For a monochromatic wave $E = E_0 \sin \omega_0 t$ we have $D = \varepsilon_0 \varepsilon(\omega_0) E$ and $U(t) = 1/2\, \varepsilon_0 \varepsilon(\omega_0) E^2(t) + C$ where C is an arbitrary constant: this is therefore not a meaningful expression for the field energy. We must consider a quasi-monochromatic wave, and we assume that $t \to -\infty$ $E \to 0$, i. e. $E = E_0(t) \exp(-i\, \omega_0 t)$ where $E_0(t \to -\infty) \to 0$.

E_0 is slowly varying in the sense that a Fourier transform of E_0 will include only components around $\omega \sim a$ where $a \langle\langle \omega_0$. See ref. 5. Since E is now complex, we can write (49) as

$$\frac{dU}{dt} = \frac{1}{4}\left(E\frac{dD^*}{dt} + E^*\frac{dD}{dt}\right) \tag{50}$$

since the products $E \cdot \dot{D}$ and $E^x \cdot \dot{D}^x$ are vanishing when averaged over time. Keeping the first terms only in a Taylor expansion we obtain[5)]

$$\frac{dD}{dt} = -i\omega\varepsilon_0\varepsilon E + \varepsilon_0\frac{d\omega\varepsilon}{d\omega}\frac{dE_0}{dt}\exp(-i\omega t) \tag{51}$$

(We omit the suffix 0 to ω). This relation can be considered a good approximation for small amplitude oscillations in weakly dispersive media. Ref. 18 considers the inclusion of higher-order terms. Substituting (51) in (50) we get

$$\frac{dU}{dt} = \frac{1}{4}\varepsilon_0\frac{d\omega\varepsilon_1}{d\omega}\frac{d}{dt}|E_0|^2 + 2\omega\varepsilon_0\varepsilon_2|E_0|^2$$
$$+ 2\varepsilon_0\frac{d\omega\varepsilon_2}{d\omega}\left(E_2\frac{dE_1}{dt} - E_1\frac{dE_2}{dt}\right) \tag{52}$$

where $E_0 = E_1 + iE_2$. Integrating (52) with respect to t, we obtain three terms. The *first* accounts for the usual approximation to the energy density (obtained by neglecting ε_2 for the frequency in question), the second term accounts for the dielectric losses. The third term cannot uniquely be attributed to either wave energy or losses, it is moreover dependent on the "path" traced out by the E field in a complex E-plane during time, i. e. it is dependent on the "history" of the E field. (Note the similarity to Gibbs' paradox). This means that we cannot define the energy density in a thermodynamical sense[x)]. If, however, $\frac{\partial \omega\varepsilon_2}{\partial\omega} = 0$ for the frequency in question, we can identify the first term in (52) as the wave energy provided the approximative equation (51) is valid. ($\varepsilon_2(k, \omega_0) = 0$ is not a sufficient condition).

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x) The author would like to thank J. Høg, M. Sc., for suggesting these arguments.

If $\frac{\partial \omega \varepsilon_2}{\partial \omega}$ is small, we may still neglect the last term in (52) although this implies the assumption that nature is "well behaved" so the term $(E_1 \frac{dE_2}{dt} - E_2 \frac{dE_1}{dt})$ is small. Considering (18) we note that $\varepsilon_1 - \Lambda_\omega$ can be viewed as a crude measure for the importance of the last term in (52). (18) does not allow a more definite statement so we turn to some specific examples.

If f(v) has a plateau $f'(v) = 0$ and (42) or (43) is satisfied, then ε_2 and $\frac{\partial \varepsilon_2}{\partial \omega}$ are both zero so the concept of wave energy is meaningful.

If f(v) has a local minimum and (42) or (43) (and therefore the Penrose criterion) is satisfied, then $\varepsilon_1 = \varepsilon_2 = 0$, but $\frac{\partial \varepsilon_2}{\partial \omega}$ may be different from zero so we cannot define the wave energy in this case. We consider it interesting that these two cases enter (46) equally, but appear different in principle according to these arguments.

If f(v) has a local minimum and the Penrose criterion for marginal stability is satisfied, but only undamped and no unstable solutions exist, then $f''(V_{min}) = 0$. This follows from (41) using the conditions for marginal stability, namely $\gamma_k = 0$ and $(\partial \gamma_k / \partial k)_{\omega = \Omega} = 0$ where $\varepsilon_2(k, \omega) = -\pi f'(\omega/k)\omega_p^2/k^2$ according to (7). Conversely a necessary condition for growing modes is $f''(V_{min}) > 0$. In this case $\frac{\partial \omega \varepsilon_2}{\partial \omega} = 0$ for $\omega = k \cdot V_{min}$, so a local minimum and a plateau for f(v) are analogous as far as the definition of wave energy is concerned.

In the case of growing or damped waves we can only define an energy if growth or damping rates are small. Since such a definition is not strictly correct[5], we shall restrict ourselves to the requirement that $\frac{\partial \omega \varepsilon_2}{\partial \omega}$ should be vanishing, but not necessarily zero. Note that this assumption was not used when obtaining (41): we assumed only that ε_2 and $\frac{\partial \varepsilon_2}{\partial \omega}$ were of the same order.

6. Discussion and Conclusions

In the preceding sections we have shown some general relations concerning dielectric functions and a few applications of these relations. In some cases we can obtain equivalent diagrams for ε in terms of electrical networks. For instance equation 10 is equivalent to

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = 0 \quad , \quad \omega_0^2 = \frac{1}{LC} \tag{53}$$

describing a series connection of a self-inductance L, resistance R, and capacitance C, while q is the charge on the capacitance. In such cases additional information can be obtained by means of network analysis (Forster's theorem etc.). It should be noted, however, that (10) and (53) are not time reversible, while this is true for the Vlasov equation. Landau damping cannot be adequately described by resistive damping. Alternatively it has been suggested[19] that a network of very many undamped resonating circuits would exhibit resistive properties in its transient response to an initial perturbation very much similar to the effect of Landau damping. It should be noted that such a network, although without resistive components, will exhibit a fluctuating voltage on account of thermal excitations. However, an electric network equalizing (7) would contain infinitely many lumped parameters (even for fixed k), and in this case network theory is no longer simple.

We shall now consider some of the reservations in the former sections.

Because of assumption II the dielectric function for electron oscillations in a cold plasma

$$\varepsilon(\omega) = 1 - \left(\frac{\omega_p}{\omega}\right)^2 \tag{54}$$

was not included in the relations shown previously. Note that (54) can be obtained as the real part of (9) in the limit $g \to 0$ and $\omega_o = 0$. Equation 54 will satisfy (13) and (14) if we define an imaginary part, ε_2, of ε by: $\varepsilon_2 = -\pi\omega_p^2\,\delta'(\omega)$. Using the relations $x\,\delta'(x) = -\delta(x)$ and $\delta(x) = \lim_{g\to o} \frac{1}{\pi} g/(x^2+g^2)$ in (9) we obtain the same result, likewise with insertion of $f(v) = \delta(v)$ in (7). (54) should therefore properly read:

$$\varepsilon(\omega) = 1 - \left(\frac{\omega_p}{\omega}\right)^2 - i\pi\,\omega_p^2\,\delta'(\omega) \tag{54a}$$

(54) is a good approximation to (7) for very high frequencies and all $k \neq 0$. For small ω an approximate relations is $\varepsilon \simeq 1 + (kd)^{-2}$, where d is the Debye length.

For completeness we would like to include truncated distribution functions in the theory also. If $f(v) = g(v)\,\epsilon(|v| - v_o)$, where ϵ is Heaviside's step function, then

$$\varepsilon_2(k,\omega) = -\pi\frac{\omega_p^2}{k^2}\left[g'\!\left(\frac{\omega}{k}\right)\epsilon\!\left(\left|\frac{\omega}{k}\right| - v_o\right) + g\!\left(\frac{\omega}{k}\right)\left(\delta\!\left(\frac{\omega}{k}+v_o\right) - \delta\!\left(\frac{\omega}{k}-v_o\right)\right)\right] \tag{55}$$

according to (7). Insertion into (13) presents no difficulties: the δ functions give rise to the terms $(v_0 + \frac{\omega}{k})^{-1} + (v_0 - \frac{\omega}{k})^{-1}$ in ϵ_1. ϵ_1 is singular for $|\frac{\omega}{k}| = v_0$. [Conversely introduction of terms of the form $(v_0 + \frac{\omega}{k})^{-1}$ in ϵ_1 gives rise to terms $\delta(\frac{\omega}{k} + v_0)$ in ϵ_2 as verified by insertion into (14) and use of Plemely's formula (also called the Poincare - Bertrnad lemma): $P\int_{-\infty}^{\infty} (x-x')^{-1}(x'-x'')^{-1}\,dx' = -\pi^2\delta(x-x'')$]. Note that (54a) and (55) contain δ functions explicitly, i. e. not under an integral sign. Strictly speaking this has no meaning, so our assumption II is justified.

Assumption I imposed a symmetry condition on ϵ, namely $\epsilon(k, \omega) = \epsilon^*(k, -\omega)$. For (7) this meant that $f(v)$ should be symmetric with respect to some v_0, i. e. $f(v_0+v) = f(v_0-v)$. Considering for instance (1) we can include slightly asymmetric $f(v)$ also. Assume that the response to a symmetric $f_0(v_0-v)$ is known and we perturb this distribution with a small $f_1(v)$ which is symmetric also: $f_1(v_1-v)$ where $v_1 \neq v_0$. If the change in $\varphi(k, \omega)$ due to f_1 is small also (remember that even a small f_1 may change the plasma from being stable to being unstable), then

$$\varphi_{0+1} = \frac{\varrho(k,\omega)}{\varepsilon_0\, k^2}\,\frac{1}{\varepsilon^0(k,\omega) + \varepsilon^1(k,\omega) - 1} \simeq \frac{\varrho(k,\omega)}{\varepsilon_0\, k^2}\,\frac{\varepsilon^1(k,\omega)}{\varepsilon^0(k,\omega)} \tag{56}$$

where $\epsilon^1(k, \omega)$ is the dielectric function due to the perturbation alone ($f_1(v)$ inserted in (7)). $\epsilon^1(k, \omega)-1$ is small. Because of the symmetry relations imposed on $f_1(v)$ the relations in the previous sections can be used for ϵ^1 also. The extension to perturbations consisting of two or more $f_i(v)$ offers no difficulties.

We mentioned that $\epsilon(k, \omega)$ often exhibits singularities for $(k, \omega) = (0, 0)$. We want to point out that this singularity is of pure mathematical origin and not a physical characteristic. It is due to the neglec of a collision term in the Vlasov equation. The Saha equation ensures that there will always be some neutrals present in a plasma in thermal equilibrium. No matter how small the neutral density is, collisions will always become important for sufficiently long wavelengths. Infinitely long wavelengths (corresponding to $k = 0$) will always be sufficiently long. When collisions are important, we can introduce a linearized collision term of the right-hand side of the Vlasov equation of the form $-\tau f_1$, where τ is the collision frequency. The normal mode solution (44) should then be multiplied by $\exp(-\tau t)$, while

$$\varepsilon(k,\omega) = 1 - \left(\frac{\omega_p}{k}\right)^2 \int_{-\infty}^{\infty} \frac{f'(v)}{v - \frac{\omega}{k} - i\frac{\tau}{k}} dv \tag{57}$$

This collision model may turn out to be insufficient in many cases because the number of particles is not conserved locally, only on average. This difficulty can be avoided by using the collision term suggested by Bhatnagar et al.[20]

$$\left[\frac{\partial f_1}{\partial t}\right]_{coll.} = -\tau \left[f_1 - n_1 f\right] \tag{58}$$

Note that this formulation does not allow for temperature fluctuations. In this case we find normal mode solutions of the form

$$f^{k,\omega} \exp(i(kx - \omega t)) \exp(-\tau t)$$

where

$$f^{k,\omega} = P \frac{\left(\frac{\omega_p}{k}\right)^2 f' - i\frac{\tau}{k} f}{v - \frac{\omega}{k}} + \left(1 - P\int_{-\infty}^{\infty} \frac{\left(\frac{\omega_p}{k}\right)^2 f'(v) - i\frac{\tau}{k} f(v)}{v - \frac{\omega}{k}} dv\right) \delta\left(v - \frac{\omega}{k}\right) \tag{59}$$

Simple calculations show that the dielectric function is given by

$$\varepsilon(k,\omega) = \left(1 - \int_{-\infty}^{\infty} \frac{\left(\frac{\omega_p}{k}\right)^2 f'(v) - i\frac{\tau}{k} f(v)}{v - \frac{\omega}{k} - i\frac{\tau}{k}} dv\right)\left(1 + i\frac{\tau}{k}\int_{-\infty}^{\infty} \frac{f(v)}{v - \frac{\omega}{k} - i\frac{\tau}{k}} dv\right)^{-1} \tag{60}$$

As mentioned earlier the use of a constant collision frequency implies the assumption that the duration of a collision is vanishing[7]. This is obviously not true for collisions between charged particles. (57) and (60) are therefore useful only when considering the effect of neutrals.

In order to avoid the influence of collisions we should consider geometries of a characteristic lenght, L, much smaller than the mean free path. We can for instance consider a plasma confined between two plane, parallel condenser plates. Assume that the distance between the plates is L and the area, A, is sufficiently large so that we can neglect boundary effects (the plates should be heated in order to maintain thermal equilibrium). The fluctuating voltage between the two terminals due to thermal excitations is then given by a superposition of (2)

$$\langle \varphi^2(\omega)\rangle = \sum_{k_n} \frac{\varkappa T}{A\varepsilon_0 k_n^2}\, \frac{2}{\omega}\, \frac{\varepsilon_2(k_n,\omega)}{\varepsilon_1^2(k_n,\omega) + \varepsilon_2^2(k_n,\omega)} \qquad (61)$$

where $k_n = n\frac{\pi}{L}$, $n = \pm 1, \pm 2, \ldots$, at least in a linear theory where the excitation of one mode is independent of the excitation of another. Conversely measurements of the thermal fluctuations for different values of L could give information on the non-linear wave-coupling mechanism.

Finally we mention that correlation of charged particles has been neglected altogether. The simplest, but still labourious, way to take this into account is to include a Fokker - Planck collision term

$$\left[\frac{\partial f}{\partial t}\right]_{coll} = -\frac{\partial}{\partial v_i}\, \frac{\langle \Delta v_i\rangle}{\Delta t} f(v) + \frac{1}{2}\, \frac{\partial^2}{\partial v_i \partial v_j}\, \frac{\langle \Delta v_i \Delta v_j\rangle}{\Delta t} f(v) \qquad (62)$$

See for instance ref. 21. In order to keep this term small (so we can speak of a truly collision-free plasma) clearly the first and second derivatives of f(v) should be small. This implies that both ε_2 and ε_1-1 should be small according to the relations obtained earlier. A truncated distribution must clearly be excluded. For instance in a single-ended Q-machine one could expect the ion stribution to be a truncated Maxwellian provided the plate temperature and work function are constant. The measured distribution is generally close to a Maxwellian as is to be expected considering (62).

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